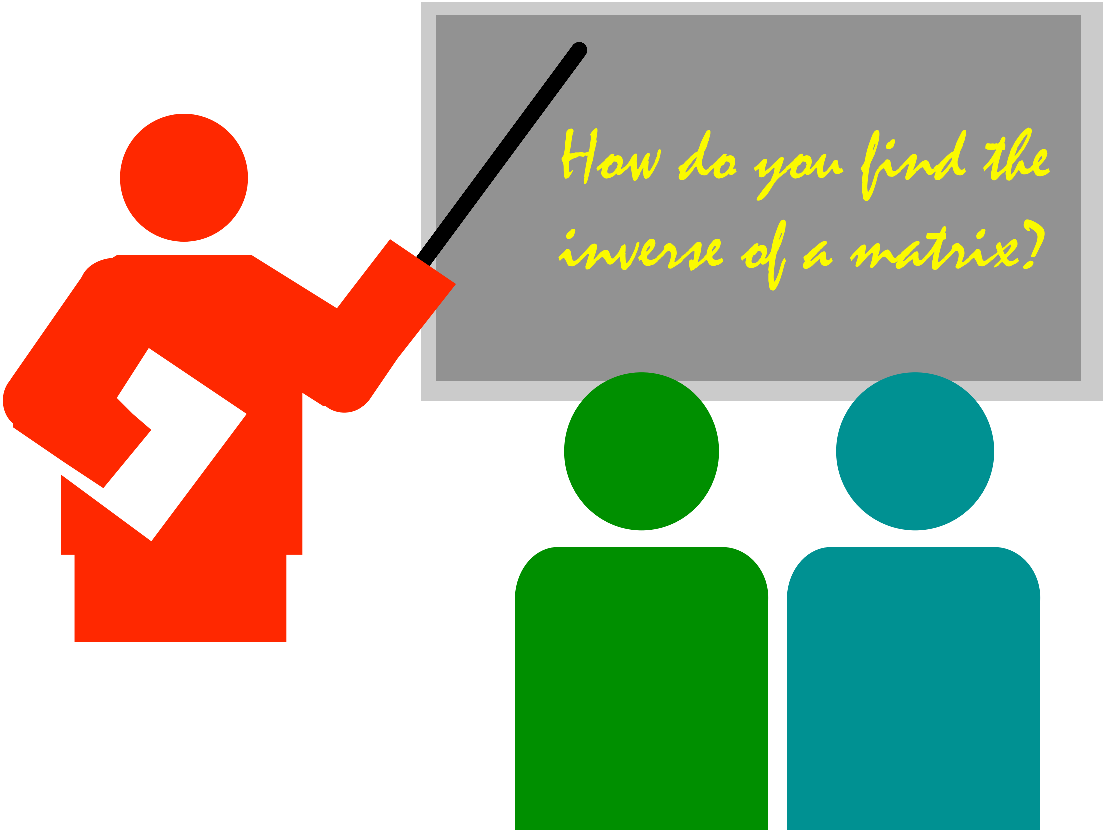
How do you find the inverse of a matrix?

Review

$$2x + 4y + 5z = 1$$
$$x + y + 2z = 2$$
$$3x + 6y + 8z = 3$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 5 & 1 \\ 1 & 1 & 2 & 2 \\ 3 & 6 & 8 & 3 \end{array}\right]$$

Need to convert this 2 to 1

$$\begin{aligned} 2x + 4y + 5z &= 1 \\ x + y + 2z &= 2 \\ 3x + 6y + 8z &= 3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 5 & 1 \\ 1 & 1 & 2 & 2 \\ 3 & 6 & 8 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 4 & 5 & 1 \\ 3 & 6 & 8 & 3 \end{array}\right]$$

Example (continued)

$$2x + 4y + 5z = 1$$
$$x + y + 2z = 2$$
$$3x + 6y + 8z = 3$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 5 & 1 \\ 1 & 1 & 2 & 2 \\ 3 & 6 & 8 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 4 & 5 & 1 \\ 3 & 6 & 8 & 3 \end{array}\right]$$

$$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 2 & 1 & -3 \\ 0 & 3 & 2 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 2 & 1 & -3 \\ 0 & 3 & 2 & -3 \end{array}\right]$$

Make into a 1

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & \frac{1}{2} & \frac{-3}{2} \\ 0 & 3 & 2 & -3 \end{array}\right]$$

Make these entries into zeros

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & \frac{1}{2} & \frac{-3}{2} \\ 0 & 3 & 2 & -3 \end{array}\right]$$

$$\xrightarrow[R_3 - 3R_2]{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} \end{array}\right]$$

Make into a 1

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} \end{array}\right]$$

$$\xrightarrow{2R_3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Make these entries into zeros

$$\xrightarrow[R_2 - \frac{1}{2}R_3]{R_1 - \frac{3}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & \frac{7}{2} \\ 0 & 1 & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Make these entries into zeros

$$\xrightarrow[R_2 - \frac{1}{2}R_3]{R_1 - \frac{3}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x = -1$

$y = -3$

$z = 3$

Matrix inverses

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix inverses

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In this case *B* is called the inverse of *A*. We write $B = A^{-1}$.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

How do we find the inverse of A?

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

How do we find the inverse of *A*?

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

How do we find the inverse of *A*?

Make these entries into zeros

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

We want zeros in these 2 spots

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

This must be made into 1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

We want zeros in these 2 spots

$$\xrightarrow{R_2 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Check by multiplying them together and observing that you get the 3x3 identity matrix.

Connection between finding a matrix inverse and solving a system of equations:

Example

$$x - z = 3$$

$$2y - 2z = 2$$

$$2x + z = 3$$

Connection between finding a matrix inverse and solving a system of equations:

Example

$$x - z = 3$$
$$2y - 2z = 2$$
$$2x + z = 3$$

coefficient matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix}$$

$$x - z = 3$$
$$2y - 2z = 2$$
$$2x + z = 3$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

These give the same information

$$x - z = 3$$

$$2y - 2z = 2$$

$$2x + z = 3$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

These give the same information

$$\text{inverse of coefficient matrix} = \begin{bmatrix} 1/3 & 0 & 1/3 \\ -2/3 & 1/2 & 1/3 \\ -2/3 & 0 & 1/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

What happens if we multiply both sides of this equation by the inverse of the coefficient matrix?

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

What happens if we multiply both sides of this equation by the inverse of the coefficient matrix?

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 1/3 \\ -2/3 & 1/2 & 1/3 \\ -2/3 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

What happens if we multiply both sides of this equation by the inverse of the coefficient matrix?

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 1/3 \\ -2/3 & 1/2 & 1/3 \\ -2/3 & 0 & 1/3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$x = 2$

$y = 0$

$z = -1$

Summary

$$x - z = 3$$
$$2y - 2z = 2$$
$$2x + z = 3$$

coefficient matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 0 & 1 \\ -2/3 & 1/2 & 1/3 \\ -2/3 & 0 & 1/3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$x = 2$
$y = 0$
$z = -1$